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THE HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

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I.

A. THE PURPOSE OF ZENO'S ARGUMENTS.

Introduction. No questions on the foundations of mathematics are as old and of such perennial interest, reaching into the most recent speculations on the philosophy of mathematics, as are Zeno's arguments on motion. Zeno flourished in the fifth century before Christ, but only recently has G. Cantor's *Mengenlehre* been applied to the fuller elucidation of Zeno's paradoxes. The history of these paradoxes is largely the history of concepts of continuity, of the infinite and infinitesimal.

There has been great difference of opinion as to the exact nature and purpose of Zeno's arguments. None of Zeno's writings have come down to us. We know of his tenets only through his critics and commentators—Plato, Aristotle and Simplicius. Plato was born about 60 years after Zeno, Aristotle about 100 years after. Simplicius lived nearly 1,000 years after Zeno. Plato does not reproduce Zeno's arguments,[1] but discusses their purpose, which was "to protect the arguments of Parmenides against those who make fun of him and seek to show the many ridiculous and contradictory results which they suppose to follow from the affirmation of the *one*"; Zeno argues that "there is no *many*," he "denies plurality."

Aristotle's Interpretation of Zeno's Arguments. Aristotle gives in very compressed form the arguments on motion, as they were handed down to him, in these words:[2]

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[1] Plato, *Parmenides*, 127 D., Jowett's translation.

[2] Aristotle, *Physics*, VI, 9. See Carl Prantl's edition, in Greek and German, Leipzig, 1854. We omit the latter part of Prantl's version of Zeno's fourth argument against motion. The text is defective. Later we give Burnet's elaboration of the proof, which embodies the most probable conjecture of what Aristotle originally said.

"Zeno reasons here incorrectly; for, he says that everything, when in a uniform state, is continually either at rest or in motion, and that a body moving in space is continually in the *Now* [the instant], hence the arrow in its flight is at rest. But this is false, for the reason that time is not composed of individual, indivisible *Nows*, as also no other quantity is so composed. There are four proofs advanced by Zeno against motion, which present many difficulties to those who try to refute them. The first is the one on the impossibility of motion, on the ground that a thing moving in space must arrive at the mid-point before it reaches the end-point. We have gone into the details of this matter in our previous discussion. The second is the so-called Achilles; it consists in this that in a race the faster cannot overtake the slower; for, the pursuer must always first arrive at the point from which the one pursued has just departed, so that the slower is necessarily always a small distance ahead. But this is the same argument as that of bisection and differs from that merely in this, that the distance added is not divided quite into halves. That the slower is not overtaken follows from this argument, but it rests upon the same assumption as the bisection (for in both arguments it is stated that a thing cannot reach the end-point, since the quantity is divided in some manner. However, this second argument has the additional contention, that, in a race, even the most rapid cannot overtake the slowest and the refutation must therefore be the same. The claim that the one in the lead cannot be overtaken is false. To be sure, in the moment when he has the lead, he is not overtaken. Nevertheless he is overtaken; Zeno merely admits that the pursuer completely passes over the entire distance. These are two of his proofs; the third is the one referred to above, that the moving arrow is at rest. It is based on the assumption that time is made up of the individual *Nows*. If this is not admitted, then the conclusion does not follow. The fourth is in regard to equal bodies which move on a track parallel to other bodies of equal size but moving in opposite directions, namely the first moving thither from the end of the track, the second moving hither from the middle of it with the same speed. From this he thought that he must conclude that the half time must be equal to its double. The fallacy lies in the claim that when a body moves parallel to one in motion, with the same speed as it does move, passes one that is at rest, the time of passing is the same in both cases. This is false."

For greater clearness we repeat Zeno's arguments in the expanded form given by Burnet,[1] which is a free paraphrase of Aristotle's statements. We shall find it convenient, for future reference, to use the names "Dichotomy," "Achilles," "Arrow," and "Stade" for the four arguments against motion, respectively.

1. "DICHOTOMY": You cannot traverse an infinite number of points in a finite time. You must traverse the halt of any given distance before you traverse the whole, and the half of that again before you can traverse the whole, and the half of that again before you can traverse it. This goes on *ad infinitum*, so that (*if space is made up of points*) there are an infinite number in any given space, and it cannot be traversed in a finite time.
2. "ACHILLES": The second argument is the famous puzzle of Achilles and the tortoise. Achilles must first reach the place from which the tortoise started. By that time the tortoise will have got on a little way. Achilles must then traverse that, and still the tortoise will be ahead. He is always nearer, but he never makes up to it.
3. "ARROW": The third argument against the possibility of motion *through a space made up of points* is that, on this hypothesis, an arrow in any given moment of its flight must be at rest in some particular point.[2]

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[1] J. Burnet, *Early Greek Philosophy*, 1892, pp. 322 ff.

[2] A different version of the "arrow" is given by Diogenes Laertius, IX, 72, who lived 500 years after Aristotle, probably about 200 A.D.: "That which moves can neither move in the place where it is, nor yet in the place where it is not." In expanded form this argument is given by William Minto, professor of logic in the University of Aberdeen, in his *Logic, Inductive and Deductive*, London, 1893, p. 224, and by W. R. Royce Gibson in his *The Problem of Logic*, London, 1908, p. 290, as follows:

"If a body moves, it must move either where it is or where it is not." "But a body cannot move where it is; neither can it move where it is not."

"Therefore, it cannot move at all; i. e. motion is impossible."

4. "STADE": Suppose three parallel rows of points in juxtaposition, as in Fig. 1.

FIG. 1.

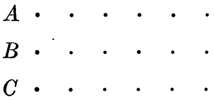
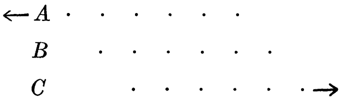


FIG. 2.



One of these (B) is immovable, while A and C move in opposite directions with equal velocity so as to come into the position represented in Fig. 2. The movement of C relatively to A will be double its movement relatively to B, or, in other words, any given point in C has passed twice as many points in A as it has in B. It cannot, therefore, be the case that an instant of time corresponds to the passage from one point to another.

Tannery's Interpretation of Zeno's Arguments. As reported by Aristotle and Simplicius, Zeno's arguments are fallacies. That Zeno's reasoning was wrong has been the view universally held since the time of Aristotle down to the middle of the nineteenth century. During these many centuries the efforts of philosophers and mathematicians on this matter were to explain the exact nature of Zeno's blunders. More recently the opinion has been advanced that Zeno was incompletely and incorrectly reported, that his arguments were turned away from their true purpose by the Sophists who used them in advancing skepticism and the denial of knowledge, and that Aristotle described them as modified by the Sophists. Three great leaders in the interpretation of Greek thought, Cousin, Grote and P. Tannery, have construed Zeno's arguments as serious efforts, conducted with logical rigor. Cousin[1] maintained that Zeno successfully opposed the idea of *multiplicity devoid of all unity*. Grote[2] held a similar view. Tannery[3] argued that Zeno opposed the idea that *a point is unity in position*. Zeller[4] rejects all three explanations, mainly on the ground that they do not find support in the extant writings of Greek philosophers. It is also true that no Greek account definitely refutes any of the three interpretations. This lack of exact and detailed information is deplorable. Tannery's resuscitation of Zeno's arguments deserves our attention because of the internal coherence imparted to them. According to Tannery, Zeno did not deny motion, but wanted to show that motion was impossible under the conception of space as the sum of points. Tannery does not battle against the traditional statement that Zeno argued against plurality; he accepts Plato's general explanation, but differs from him and others on the precise nature of this plurality. According to Tannery it was not the ordinary notion that Zeno combated, according to which two lambs

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This version of the "Arrow" must be rejected for two reasons: First, it occurs for the first time about 700 years after Zeno and is for that reason unreliable; second, there is no kernel to the argument. As Gibson says, it amounts to this: "If a body moves, it must move under conditions which render motion impossible."

[1] *Fragments philosophiques*, par M. Cousin. 5ème éd., Paris, 1865, p. 69.

[2] George Grote, *Plato*, Vol. I, 3d ed., London, 1875, pp. 100–104.

[3] Paul Tannery, "Le concept scientifique du continu. Zénon d'Elée et Georg Cantor," *Revue philosophique de la France et de L'Etranger*, X année, T. XX (1885), pp. 385–410; Paul Tannery, *Science hellène*, Paris, 1887, pp. 247–261.

[4] E. Zeller, *Die Philosophie der Griechen*, 1. Theil, 1. Hälfte, 5. Aufl., Leipzig, 1892, pp. 591–604.

are not one lamb, but a special notion of the Pythagoreans. Zeno's master, Parmenides, was attacked (says Tannery) by the Pythagoreans, and Zeno stepped into the conflict by battling against the mystical Pythagorean idea of a mathematical point—a point defined as *unity having position.* This Pythagorean definition is mentioned by Aristotle. Tannery interpreted this definition as signifying that a solid is the sum of points, just as a number is the sum of units. But such an idea is false. A point, mathematically speaking, is not unity or 1; it is a pure zero or 0. This interpretation of the phrase *unity having position* attributes to Zeno the grasp of an abstract concept, namely that of a point, destitute of length, breadth and thickness. That it is not unreasonable to ascribe to Zeno such abstractions seems evident from the following passage in Aristotle:[1]

> "If the absolute unit is indivisible it would be, according to Zeno's axiom, nothing at all. For that which neither makes anything larger by its addition, nor makes anything smaller by its subtraction, is not one of the things that are, since it is clear that what *is* must be a magnitude, and, if a magnitude, corporeal, for the corporeal has being in all dimensions. Other things, such as the surface and the line, when added in one way make things larger, when added in another way do not; but the point and the unit do not make things larger however added."

It is hard to make out how much of this is the thought of Aristotle, and how much of it is Zeno's. Yet there would be no occasion to mention Zeno, had he no share in it.

It is believed by many critics that Zeno gave his arguments in the form of dialogue. Acting upon this view, Tannery entered upon the reconstruction of Zeno's arguments from the compressed passages handed down to us. Consider the following argument of Zeno on divisibility, as stated by Simplicius:[2]

> "If that which is, has no magnitude it could not even be. Everything that truly is must needs have magnitude and thickness, and one part of it must be separated from another by a certain interval. And the same may be said of the next smaller part; it too will have magnitude, and a next smaller part. As well say this once for all as keep repeating it forever. For there will be no such part that could serve as a limit. And there will never be one part save in reference to another part. Thus, if the many have being, they must be both large and small—so small as to have no size at all, and so large as to be infinite."

Tannery's reconstruction of this passage is as follows:

A Pythagorean adversary claims that a finite quantity can be regarded as the sum of indivisible parts.

Zeno presents the first part of the dilemma resulting therefrom, thus: Admitting, as both of us do, that a quantity is infinitely divisible by continued bisection, it is evident that the parts become smaller and smaller. Hence, if there is a last term, it is 0. But the sum of such indivisible terms 0 is only 0. Hence the quantity has no magnitude.

But, says his adversary, why may the indivisible parts not be different from 0 and have magnitude?

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[1] Aristotle, *Met.*, II, 4, 1001 b 7; translation taken from C. M. Bakewell, *Source Book in Ancient Philosophy*, New York, 1907, p. 23.

[2] Simpl. 140, 34 [R. P., 1050, Fr. 2 in Diels' arrangement]; C. M. Bakewell, *op. cit.*, p. 22.

Then Zeno presents the second part of the dilemma: If the indivisible parts have magnitude, and are infinite in number, the sum of these parts must be infinite.

Consequently, a finite quantity cannot be regarded as the sum of indivisible parts.

This explanation of Zeno's argument places Zeno certainly higher as a logician than does the old explanation which charged Zeno with inability to see that, if $xy = c$, x can increase and y simultaneously decrease in such a way that their product remains the same.

Now, let us see how Tannery applies *a point as unity in position* to the resuscitation of Zeno's arguments on motion. Tannery presents them in the form of a double dilemma.

The first argument, the "Dichotomy," involves matters which we have considered above, in connection with infinite division. As long as space is assumed to be made up of indivisible parts, the infinite number of parts, admitted by both contestants to result from continued bisection, cannot all be passed over in a given time.

The adversary may now present the point advanced by Aristotle, that the bisection is not carried on to actual infinity, but only to a potential infinity, and may therefore be run over in a finite time.

Zeno replies by stating the "Achilles," which does not involve bisection and in which the time-interval is subdivided in much the same way as the space-interval.

The adversary then takes the position that he has admitted too much. Finite time, he claims, is capable of division into an infinity of parts. Is there not a sum of instants? May there not correspond an instant to each successive position?

Against this Zeno directs his last two arguments, which constitute a second dilemma. At each instant the flying arrow occupies a fixed position. But occupying a fixed position at a given instant means that it is at rest that instant. Hence the arrow is at rest every instant of its flight.

The adversary explains that when saying that time was the sum of instants, he did not mean that each instant should apply to a fixed position of the arrow, but rather to the passage from each position to the next following position.

Here Zeno advances his "Stade" as his fourth argument. He shows that the demand of his adversary cannot be granted, because it would make all motions equal.

A motion from a point A (see Fig. 1 and Fig. 2) to the next point on the left requires one instant.

A motion from a point C to the next point on the right requires the same instant.

Hence A moves relatively to C twice as fast as relatively to B.

It is therefore not the passage from one point to the next that corresponds to the instant, for it would then follow that one is equal to its double.

Tannery's explanation of the four arguments, particularly of the "Arrow" and "Stade," raises these paradoxes from childish arguments to arguments with conclusions which follow with compelling force. It does not place Zeno in the position of being ignorant of the most simple ideas of relative motion; it exhibits Zeno as a logician of the first rank.

Tannery's conclusions have been strongly supported by G. Milhaud,[1] but opposed by other French writers and by Zeller.

CENTERS OF SIMILITUDE OF CIRCLES AND CERTAIN THEOREMS ATTRIBUTED TO MONGE. WERE THEY KNOWN TO THE GREEKS?

By RAYMOND CLARE ARCHIBALD, Brown University.

One of the most noticeable characteristics of French, German and Italian, as opposed to American, texts on elementary geometry is the emphasis laid on broad underlying principles. How many American high-school graduates could give one any idea of the theory of similitude of plane and solid figures? How many teachers realize the importance of this far reaching theory in the solution of geometrical problems, or are familiar with the equivalent of Petersen's excellent exposition?[2] At all events it seems well worth while to draw attention to some simple results in the theory, and to put on record their historical setting. The theorems attributed to Monge, which I propose to discuss, involve the centers of similitude of circles (spheres).

The centers of similitude of two circles (spheres), whose centers are A and B, are the points which divide AB internally, at C, and externally, at D, in the ratio of the radii, r_a, r_b ($r_a \geqq r_b$);

$$AC : CB = AD : BD = r_a : r_b.$$

The circles (spheres) may be situated in any fashion. If they are tangent (internally or externally), the point of tangency is a center of similitude. If concentric, we may, perhaps, say that either A, B, C, D coincide or else A, B, C coincide while D is indeterminate. If they are equal and non-concentric, D is at infinity[3] and C bisects AB.

Lines joining the ends of parallel radii pass through a center of similitude, and common tangent lines (planes), when they exist, also pass through such a center. Conversely, if through a center of similitude, D (or C), of two circles a line be

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[1] G. Milhaud, "Le concept du nombre chez les Pythagoriciens et les Éléates," *Revue de métaphysique et de morale*, I, Paris, 1893, p. 141.

[2] J. Petersen, *Methods and Theories for the Solution of Problems of Geometrical Constructions*, Copenhagen, 1879, pp. 22 ff. This is an English edition of the remarkable Danish original. There are also French, German, Italian, Hungarian and Russian translations making in all some 15 editions. Because of its many notable qualities, this work stands preeminent in its special field.

[3] This is, of course, more an idea of *projective*, than of *elementary*, geometry.